

# Fast and Accurate Phase Unwrapping

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## Abstract

More and more speech technology and signal processing applications make use of the phase information. A proper estimation and representation of the phase goes inextricably along with a correct phase unwrapping, which refers to the problem of finding the instance of the phase function chosen to ensure continuity. This paper proposes a new technique of phase unwrapping which is based on two mathematical considerations: *i*) a property of the unwrapped phase at Nyquist frequency, *ii*) the modified Schur-Cohn's algorithm which allows a fast calculation of the root distribution of polynomials with respect to the unit circle. The proposed method is compared to five state-of-the-art phase unwrappers on a large dataset of both synthetic random and real speech signals. By leveraging the two aforementioned considerations, the proposed approach is shown to perform an exact estimation of the unwrapped phase at a reduced computational load.

**Index Terms:** Phase unwrapping, Phase Information, Group Delay, Root Distribution

## 1. Introduction

The use of the phase information has gained more and more interest in various areas of speech processing. This is motivated by the fact that the phase is expected to convey information that is complementary to the conventional features used in the great majority of current approaches. These features are generally derived from a representation of the spectral envelope and consequently exploit only the amplitude component of the Fourier transform, and discard its phase counterpart.

Several studies have therefore targeted the incorporation of phase information within numerous voice technology applications [1]. In speech synthesis, a proper modeling of the phase information allows an improved reconstruction of the excitation signal and reduces the buzziness of the generated speech [2, 3]. In speech analysis, the phase information is required in various tasks such as complex cepstrum calculation [4], glottal flow estimation [5, 6], glottal closure instant detection [7], speech polarity determination [8] or maximum voiced frequency estimation [9]. In speech recognition, several representations based on the group delay function (defined as the first derivative of the phase) have been proposed and were reported to provide an enhanced robustness [10, 11, 12]. The phase information was also found to be useful for other purposes such as speaker identification [13, 14, 15] or voice disorder detection [16, 17, 18]. Note that its use is not restricted to the speech signal, and covers other types of signals such as seismic [19], electroencephalographic [20] or image data [21]. This paper however focuses on phase unwrapping for one-dimensional signals.

Across all techniques, a proper estimation of the phase is required. The phase of the Discrete Time Fourier Transform (DTFT) is generally ambiguous as a multiple of  $2\pi$  can be added at any frequency without influencing the resulting decomposition in complex exponentials [4]. The resulting phase representation might therefore exhibit irrelevant jumps. Phase unwrapping refers to the problem of finding the phase function for which the additive integer multiples of  $2\pi$  at each frequency are chosen to ensure its continuity.

This paper proposes a new method of phase unwrapping which is both fast and accurate. It is structured as follows. Section 2 briefly discusses the link between the location of the zeroes of the z-transform and the difficulties in phase unwrapping. Approaches presented in the literature and the proposed method are respectively described in Sections 3 and 4. Our experimental results are discussed in Section 5 and Section 6 finally concludes the paper.

## 2. Distributions of the Zeros and Phase Unwrapping Difficulties

Let us consider a finite-length discrete real signal  $x(n)$  of  $N$  samples. Its z-transform can be written as:

$$X(z) = \sum_{n=0}^{N-1} x(n)z^{-n} = x(0)z^{N-1} \prod_{i=1}^{N-1} (z - z_i), \quad (1)$$

provided that  $x(0)$  is non-zero, and where  $z_i$  are the zeros of the signal. The DTFT can be expressed through this representation when  $z = e^{j\omega}$ , where  $\omega$  is the angular frequency. Let us now consider a particular zero  $z_m$ . As illustrated in Figure 1, the phase jump induced by this zero between two consecutive frequencies  $\omega_k$  and  $\omega_{k+1}$  of the DFT can be substantial if  $z_m$  is located in the vicinity of the unit circle (UC) in the z-plane. This will make any DFT-based phase unwrapping more difficult as a rather dense resolution of the  $\omega_k$  on the UC will be required (i.e. a high number of DFT samples will be necessary).

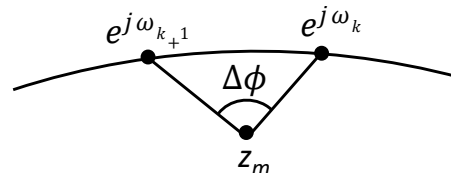


Figure 1: Main difficulty with DFT-based phase unwrapping: due to zeros close to the unit circle, substantial phase discontinuities may occur if the DFT resolution is not sufficient.

T. Drugman is now with Amazon. This work was entirely carried out when he was with Toshiba Cambridge Research Laboratory.

The main issue is that the zeros of random polynomials and those of signals such as speech are known to cluster in two an-

nuli in the close vicinity and on both sides of the unit circle [22, 23]. This phenomenon gets even more pronounced when the length  $N$  of the signal increases. As an illustration, we found that for random polynomials of degree 100 and 1000, respectively 40% and 91.3% of the zeros have a radius ranging from 0.99 to 1.01. For speech segments sampled at 32 kHz, these rates are of 21.2% and 83.7%.

### 3. Existing Techniques for Phase Unwrapping

The most popular technique of phase unwrapping is based on the detection and removal of the phase discontinuities between two adjacent DFT samples [4]. When the phase jump between two consecutive DFT samples in  $\omega_k$  and  $\omega_{k+1}$  exceeds a certain threshold (usually set to  $\pi$ ), the phase in  $\omega_{k+1}$  is corrected by adding a multiple integer of  $2\pi$ . That is the method which is currently implemented in the *unwrap* function in Matlab. The advantage of this technique is that it is rather simple and fast. However, we know from Section 2 that the phase may vary rapidly if some zeros are close to the UC. A sufficient resolution of the DFT samples on the UC will therefore be required to avoid unwrapping errors. This method will be referred to as the Discontinuity Detection (DD) method in the following. An alternative to DD is to start from a poor resolution (i.e. low number of DFT samples) and increase gradually the resolution until two consecutive unwrapped phase estimates match. Such a matching does not however guarantee that the resulting estimate is correct. This approach will be referred to as the Matching (M) method.

Some other techniques are based on a Numerical Integration (NI) of the phase derivative. Indeed, it can be shown that the phase derivative  $\phi'(\omega)$  can be expressed as [24]:

$$\phi'(\omega) = -\frac{X_R(\omega)Y_R(\omega) + X_I(\omega)Y_I(\omega)}{|X(\omega)|^2} \quad (2)$$

where  $Y(\omega)$  is the DFT of  $nx(n)$  and R and I refer to the real and imaginary parts. Here again, a high number of DFT samples might be necessary to perform the NI with a sufficient integration step size. Tribolet proposed in [25] an adaptive NI method which gradually decreases the integration step until the samples of the estimated unwrapped phase are consistent with those of the wrapped phase. Variants of this method differ by the way the NI is performed: using a trapezoidal rule [25], the two individual envelopes along with their average [26], or a piecewise polynomial interpolation [27].

Another approach is based on Polynomial Factoring (PF, [28]). The idea is to factorize  $X(z)$  and therefore find the zeros  $z_i$ . From (1), it can be demonstrated (see Section 4.1) that the unwrapped phase  $\phi(\omega)$  can be simply obtained by adding together the phase contribution of each root separately. The advantage of this method is that if the PF is performed accurately, then the unwrapped phase is correct by definition. Two main methods are possible for factoring high-degree polynomials: *i*) computing the eigenvalues of the companion matrix [29], *ii*) using Fast Fourier Transform (FFT) to create a search grid in the  $z$ -plane [30]. The first PF technique is  $O(N^3)$  in time while the second is  $O(N^2)$ . PF-based unwrapping is therefore more computationally demanding compared to DFT-based approaches (such as DD and NI) that can use the Fast Fourier Transform (FFT) algorithm which is  $O(N_{\text{FFT}} \log(N_{\text{FFT}}))$ .

Various other approaches have been proposed in the literature. A method based on an iterative computation of the cor-

responding minimum-phase sequence of  $x(n)$  was presented in [31]. The main problem of this technique is that the minimum-phase sequence is in general infinite-length and that the number of zeros outside the unit circle is assumed to be known a priori (which is not the case in practice). A method which uses  $x(n)$  directly to compute the additive factor required at each frequency sample to unwrap the phase was proposed in [32]. This technique is however highly sensitive to computation errors and is therefore limited to short sequences only. The method described in [33] isolates the location of the zeros that are close to the UC. It aims at finding the radial positions of sharp zeros with segments in the  $z$ -plane. However it performs poorly when zeros are clustered close to each other in the vicinity of the UC. Two composite algorithms were proposed in [34]. They combine PF-based phase unwrapping for zeros close to the UC, and DD or NI for those located more remotely. Finally, the technique proposed in [35] unwraps the phase by representing the unit circle image in terms of Tchebyshev polynomials.

### 4. Proposed Method

The core of the proposed method relies on two mathematical considerations: *i*) a property which connects the unwrapped phase in  $\omega = \pi$  to the number of zeros outside the UC; *ii*) the modified Schur-Cohn's algorithm which allows a fast calculation of the root distribution with respect to the UC. These two points were known from the state of the art and by exploiting them efficiently, the proposed approach achieves accurate phase unwrapping at a low computational cost.

#### 4.1. Property of the unwrapped phase at Nyquist frequency

Based on (1), the unwrapped phase  $\phi(\omega)$  at the angular frequency  $\omega$  can be expressed as:

$$\phi(\omega) = \arg(x(0)) + (-N + 1)\omega + \sum_{i=1}^{N-1} \arg(e^{j\omega} - z_i) \quad (3)$$

The unwrapped phase can be easily obtained by summing up the phase contribution from each zero  $z_i$  individually. The last factors in (3) can be seen as the arguments of vectors starting at  $z_i$  and ending in  $e^{j\omega}$ . Let us now consider the effect of each zero  $z_i$  separately on the value of  $\phi(\pi) - \phi(0)$ , depending upon the location of this zero in the  $z$ -plane. If  $z_i$  is a single real zero outside the UC then  $\phi(\pi) - \phi(0) = 0$ . However, if it is inside the UC, it takes a value of  $\pi$ . For real signals (which is the scope of this paper), complex zeros occur in conjugate pairs. If the pair of zeros is located outside the UC, then  $\phi(\pi) - \phi(0)$  is again equal to 0, while if it is inside the UC,  $\phi(\pi) - \phi(0) = 2\pi$ .

If the  $z$ -transform of the signal has no zeros on the UC, which is a common assumption [34] made to guarantee the continuity of  $\phi(\omega)$  and which holds in practice, it can be understood that the total value of  $\phi(\pi) - \phi(0)$  is connected to the distribution of the zeros with respect to the UC. Usually, it is assumed that  $\phi(0) = 0$ . From the previous discussion, it turns out that the value of the unwrapped phase at Nyquist frequency  $\omega = \pi$  can be determined as:

$$\phi(\pi) = \pi(-N + 1 + N_{IUC}) = -\pi N_{OUC}, \quad (4)$$

where  $N_{IUC}$  and  $N_{OUC}$  denote the number of zeros respectively inside and outside the UC. Note that this property is linked in mathematics to the Cauchy's residue theorem [36] and to the winding number.

## 4.2. Schur-Cohn's algorithm

Several studies have addressed the determination of the stability of linear systems based the root distribution of polynomials with respect to the UC, and this without requiring an explicit factorization of these polynomials (whose degree can be very high). These works encompass Bistritz's [37] and Schur-Cohn's algorithm [38]. Both algorithms associate to the original polynomial a new sequence of symmetric polynomials of descending degrees and a sequence of derived coefficients. The number of roots inside and outside the UC can then be determined using a very simple criterion from the sequence of derived coefficients. While these two methods have the same growth of complexity, we noticed the modified Schur-Cohn's algorithm to be faster. This method is depicted in Algorithm 1. It allows a fast calculation of the number of roots outside the UC for a polynomial  $P(z)$  of degree  $N$ . Note that in these notations, the operator  $*$  denotes the reciprocal of a polynomial.

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**Algorithm 1** The modified Schur-Cohn's algorithm [38]

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- 1) Initial condition:  $P_N = P$
  - 2) For  $j = N, N - 1, \dots, 1$ , compute  $P_{j-1}$  as follows:  

$$k_j = \frac{-P_j(0)}{P_j^*(0)}$$

$$z^{d_j} Q(z) = P_j(z) + k_j P_j^*(z)$$

$$Q(0) \neq 0 \text{ or } Q = 0$$
 If  $|k_j| < 1$ , take  $P_{j-1} = Q$   
 Else if  $|k_j| > 1$ , take  $P_{j-1} = Q^*$   
 Else if  $|k_j| = 1$  and if  $Q = 0$ , take  $P_{j-1} = P_j'$   
 Else take  $P_j = Q + \frac{1}{c} P_j + k_j c P_j^*$ , where  $c$  is an arbitrary real parameter between 0 and 1.
  - 3) the number  $N_{OUC}$  of roots outside the UC is the number of  $k_j$  satisfying  $|k_j| > 1$ .
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## 4.3. The proposed algorithm

The basis of the proposed approach is that a correct unwrapped phase can be obtained by simple FFT-based methods such as DD or (adaptive) NI, as long as the resolution of the FFT samples along the UC is sufficient (i.e. as long as the number  $N_{FFT}$  of FFT points is sufficiently high). Choosing a fixed extremely large value of  $N_{FFT}$  is possible to ensure a perfect phase unwrapping, even for signals with a large number of points  $N$ . However if  $N_{FFT}$  is prohibitively large, this will result in an excessively heavy computational load. The key idea of the proposed method is to adapt  $N_{FFT}$  depending upon the root distribution, so that the estimation is both accurate and fast.

The proposed method is detailed in Algorithm 2. It first calculates  $N_{OUC}$ , the number of zeros outside the UC, using the modified Schur-Cohn's algorithm. An initial low resolution is chosen by giving a first value to  $N_{FFT}$ , and the FFT-based estimate  $\tilde{N}_{OUC}$  of the number of zeros outside the UC is given a dummy negative value. An estimate  $\tilde{\phi}(\omega)$  of the unwrapped phase can then be obtained by using a simple method such as DD or NI (see Section 3). In the remainder of this paper, we have used DD as it is much faster compared to adaptive NI [34]. Any other FFT-based phase unwrapper could be used though. As discussed in Section 4.1, the value of the FFT-based estimate  $\tilde{N}_{OUC}$  can then be updated as  $\frac{-\tilde{\phi}(\pi) - \tilde{\phi}(0)}{\pi}$ . The result is compared to  $N_{OUC}$ . If the two values match,  $\tilde{\phi}(\omega)$  is the correct unwrapped phase  $\phi(\omega)$ . Otherwise, that means that the FFT resolution was not sufficient, and the procedure is repeated using an increased value of  $N_{FFT}$ .

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**Algorithm 2** The proposed phase unwrapping algorithm

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- 1) Compute  $N_{OUC}$  using the modified Schur-Cohn's algorithm
  - 2) Initialize  $\tilde{N}_{OUC}$  to a dummy negative value
  - 3) Initialize  $N_{FFT}$ , which defines the FFT resolution
  - 4) While  $\tilde{N}_{OUC}$  is different from  $N_{OUC}$ , repeat the following procedure:  
 Compute an estimate  $\tilde{\phi}(\omega)$  of the unwrapped phase using the DD technique  
 Update  $\tilde{N}_{OUC}$  to  $\frac{-\tilde{\phi}(\pi) - \tilde{\phi}(0)}{\pi}$   
 Double the value of  $N_{FFT}$
  - 5) The correct unwrapped phase is  $\tilde{\phi}(\omega)$
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## 5. Experiments

### 5.1. Experimental Protocol

The evaluation database consists of both synthetic random and real speech signals. They have a length of  $N = 10^\alpha$  samples, where  $\alpha$  is varied from 1 to 3.4 by steps of 0.2. The number of samples therefore ranges from 10 to 2512. Speech signals are Hamming-windowed segments taken randomly from the CMU ARCTIC database [39] of the male speaker BDL sampled at 32 kHz. The longest signals therefore correspond to 78.5 ms long segments. For each type of signal and each length  $N$ , 50,000 signals were considered.

Six phase unwrapping techniques are compared in the evaluation. For FFT-based approaches, the number of points is controlled by a parameter  $K = \frac{N_{FFT}}{N}$  and by applying zero-padding. These six techniques have been described in Sections 3 and 4 and are: the DD method using  $K = 64$  and  $K = 1024$ , the M method which iteratively doubles  $K$  until two consecutive unwrapped phase estimate match, the PF approach where the zeros of the polynomial are computed either using the eigenvalues of the companion matrix (function *roots* in Matlab) or the FFT-based search grid (function *lroots* [30]), and the proposed method. The 6 techniques are compared across two main dimensions: the accuracy via the error rate (defined as the proportion of erroneous estimates, i.e. the percentage of cases where  $\tilde{\phi}(\omega)$  and  $\phi(\omega)$  differ) and the complexity via the computational time to get an estimate (with Matlab implementations run on an Intel Core i7 3.0 GHz CPU with 16GB of RAM). To calculate the accuracy, one needs to compute the reference unwrapped phase using an exact method. For this purpose, we used the PF approach with the eigenvalues of the companion matrix.

### 5.2. Results

Regarding the accuracy, the two PF-based techniques gave by definition a perfect estimation as they use directly the root locations. The criterion used in the proposed method also guarantees a perfect unwrapping. The three other approaches however are prone to make some errors. Their performance is shown in Fig. 2. The worst technique is M, which in about 15.8% of the cases generated an error. This is because the matching criterion between consecutive estimates does not ensure a correct result. The error rate produced by DD increases with  $N$  and can be reduced by increasing  $K$  (and thus  $N_{FFT}$ ). It could even reach 0 if  $K$  were sufficiently large.

In terms of computational time, the results are presented in Figure 3. For signals with  $N > 200$  samples (which is in practice almost always the case), the two PF-based approaches are by far too expensive (as they are in  $O(N^3)$  and  $O(N^2)$ ). The FFT-based techniques (in  $O(N_{FFT} \log(N_{FFT}))$ ) are more advan-

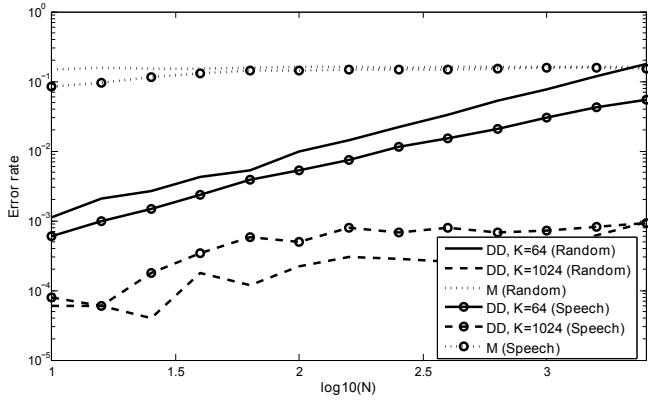


Figure 2: Error rate made by 3 phase unwrapping techniques for both random and speech signals. Note that the PF-based and the proposed methods did not produce errors.

tageous in that respect. As aforementioned, the error rate of the DD method can be reduced by increasing  $K$ . This however also goes along with a substantial increase of the computation time (see the results of DD,  $K=1024$  in Fig. 3). Interestingly, the proposed method exhibits a computational load comparable to that of the DD,  $K=64$  and M methods. Contrary to them however, it did not generate any unwrapping error.

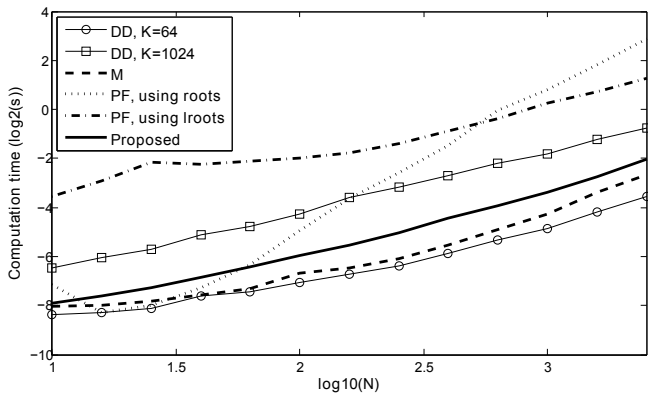


Figure 3: Computation time as a function of  $N$  for various phase unwrapping techniques.

These good computational capabilities of the proposed technique can be further analyzed by inspecting the value of  $K$  once the stopping criterion is met. The histograms for  $N = 100$  and  $N = 1000$  and for random signals are displayed respectively in Figures 4 and 5. These plots are to be compared to the fixed values 6 and 10 of  $\log_2(K)$  used in the DD technique in the evaluation (Figures 2 and 3). In the majority of cases, a lower number of FFT points was used compared to the conventional DD approach (which uses a fixed value of  $K$ ). For the signals which were more prone to errors (i.e. containing zeros very close to the UC), the proposed method automatically increased the value of  $K$  until a perfect estimation was possible (which was guaranteed via its stopping criterion).

The computational complexity analysis of the proposed approach (following the steps described in Algorithm 2) can be broken down as: *i*) step 1 (the modified Schur-Cohn's algorithm) is rather fast and is  $O(N)$ ; *ii*) step 4 is iterative (each

iteration performing a FFT in  $O(N_{\text{FFT}} \log(N_{\text{FFT}}))$  and the number of iterations depends upon whether the signal contains or not zeros in the close vicinity of the unit circle. In any case, the computational time taken by step 4 is dominated by the final value of  $K$  (hence of  $N_{\text{FFT}}$ ) whose distribution is given in Figures 4 and 5. Note that across all our experiments (covering 650,000 signals in total), convergence was reached in all cases. In the worst configurations, the convergence was slower as it implied using a rather large final value of  $N_{\text{FFT}}$ . This however concerned a minority of cases, as highlighted in Figures 4 and 5. Finally, no issues with local minima were observed, and the unwrapped phase estimated by the proposed approach always matched that using the PF technique (which is known to be exact).

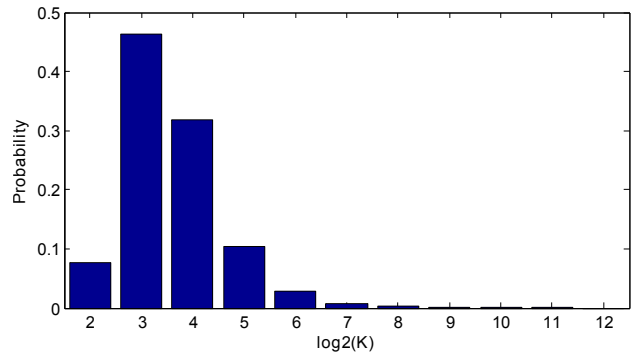


Figure 4: Histograms of  $K = \frac{N_{\text{FFT}}}{N}$  required by the proposed method for  $N = 100$ .

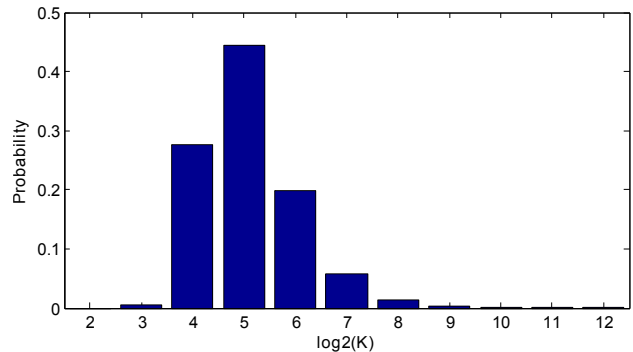


Figure 5: Histograms of  $K = \frac{N_{\text{FFT}}}{N}$  required by the proposed method for  $N = 1000$ .

## 6. Conclusions

This paper proposed a new method of phase unwrapping which exploits the link between the value of the unwrapped phase at Nyquist frequency and the number of zeros of the z-transform outside the unit circle. This latter number can be rapidly determined using fast specific algorithms such as the modified Schur-Cohn's technique. The proposed method consists of an iterative procedure whose stopping criterion guarantees a perfect phase unwrapping and which is generally achieved at a limited computational load. It was compared to 5 other state-of-the-art approaches on a large database of random and speech signals. The proposed method was shown to provide a perfect accuracy at a relatively low computational cost.

## 7. References

- [1] L. Alsteris and K. Paliwal, "Short-term phase spectrum in speech processing: a review and experimental results," *Digital Signal Processing*, vol. 17, pp. 578–616, 2007.
- [2] T. Drugman and T. Dutoit, "The deterministic plus stochastic model of the residual signal and its applications," *IEEE Trans. Audio, Speech, Lang. Proc.*, vol. 20, no. 3, pp. 968–981, 2012.
- [3] G. Degottex and D. Erro, "A measure of phase randomness for the harmonic model in speech synthesis," *Interspeech*, 2014.
- [4] A. Oppenheim, R. Schaffer, and J. Buck, "Discrete-time signal processing," *Prentice Hall, Englewood Cliffs, NJ*, 1998.
- [5] T. Drugman, B. Bozkurt, and T. Dutoit, "Causal-anticausal decomposition of speech using complex cepstrum for glottal source estimation," *Speech Communication*, vol. 53, no. 6, pp. 855–866, 2011.
- [6] G. Degottex, A. Roebel, , and X. Rodet, "Function of phase-distortion for glottal model estimation," *ICASSP*, pp. 4608–4611, 2011.
- [7] T. Drugman, M. Thomas, J. Gudnason, P. Naylor, and T. Dutoit, "Detection of glottal closure instants from speech signals: a quantitative review," *IEEE Trans. Audio, Speech, Lang. Proc.*, vol. 20, no. 3, pp. 994–1006, 2012.
- [8] I. Saratxaga, D. Erro, I. Hernez, I. Sainz, and E. Navas, "Use of harmonic phase information for polarity detection in speech signals," *Interspeech*, pp. 1075–1078, 2009.
- [9] T. Drugman and Y. Stylianou, "Maximum voiced frequency estimation: Exploiting amplitude and phase spectra," *IEEE Signal Processing Letters*, vol. 21, no. 10, pp. 1230–1234, 2014.
- [10] R. Hegde, H. Murthy, and V. Gadde, "Continuous speech recognition using joint features derived from the modified group delay function and mfcc," in *Proc. ICSLP*, 2004.
- [11] B. Bozkurt, L. Couvreur, and T. Dutoit, "Chirp group delay analysis of speech signals," in *Speech Comm.*, vol. 49, 2007, pp. 159–176.
- [12] E. Loweimi, S. Ahadi, and T. Drugman, "A new phase-based feature representation for robust speech recognition," *ICASSP*, pp. 7155–7159, 2013.
- [13] L. Wang, K. Minami, K. Yamamoto, and S. Nakagawa, "Speaker identification by combining mfcc and phase information in noisy environments," *ICASSP*, pp. 4502–4505, 2010.
- [14] T. Drugman and T. Dutoit, "On the potential of glottal signatures for speaker recognition," *Interspeech*, pp. 2106–2109, 2010.
- [15] I. Hernaez, I. Saratxaga, J. Sanchez, E. Navas, and I. Luengo, "Use of the harmonic phase in speaker recognition," *Interspeech*, pp. 2757–2760, 2011.
- [16] T. Drugman, T. Dubuisson, and T. Dutoit, "Phase-based information for voice pathology detection," *ICASSP*, pp. 4612–4615, 2011.
- [17] M. Koutsogiannaki, O. Simantiraki, G. Degottex, and Y. Stylianou, "The importance of phase on voice quality assessment," *Interspeech*, 2014.
- [18] T. Drugman, T. Dubuisson, and T. Dutoit, "On the mutual information between source and filter contributions for voice pathology detection," *Interspeech*, pp. 1463–1466, 2009.
- [19] P. Bormann and E. Wielandt, "Seismic signals and noise," *New Manual of Seismological Observatory Practice 2*, pp. 1–62, 2013.
- [20] B. Ng, N. Logothetis, and C. Kayser, "EEG phase patterns reflect the selectivity of neural firing," *Cereb Cortex*, vol. 23, no. 2, pp. 389–398, 2013.
- [21] D. Ghiglia and M. Pritt, "Two-dimensional phase unwrapping: Theory, algorithms, and software," *John Wiley and Sons, Inc.*, 1998.
- [22] I. Ibragimov and O. Zeitouni, "On roots of random polynomials," *Trans. of the American Mathematical Society*, vol. 349, no. 6, pp. 2427–2441, 1997.
- [23] B. Bozkurt, "Zeros of the z-transform (ZZT) representation and chirp group delay processing for the analysis of source and filter characteristics of speech signals," *PhD thesis, Faculte Polytechnique de Mons*, 2005.
- [24] H. Banno, L. Jinlin, S. Nakamura, K. Shikano, and H. Kawahara, "Efficient representation of short-time phase based on group delay," *ICASSP*, vol. 2, pp. 861–864, 1998.
- [25] J. Tribolet, "A new phase unwrapping algorithm," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 25, no. 2, pp. 170–177, 1977.
- [26] J. Scott, "Improving confidence in the phase unwrapping algorithm," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, vol. 32, no. 6, pp. 1254–1255, 1984.
- [27] B. Bhanu and J. McClellan, "On the computation of the complex cepstrum," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 28, no. 5, pp. 583–585, 1980.
- [28] K. Steiglitz and B. Dickinson, "Phase unwrapping by factorization," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 30, no. 6, pp. 984–991, 1982.
- [29] A. Edelman and H. Murakami, "Polynomial roots from companion matrix eigenvalues," *Math. Comp.*, vol. 64, pp. 763–776, 1995.
- [30] G. Sitton, C. Burrus, J. Fox, and S. Treitel, "Factoring very-high-degree polynomials," *IEEE Signal Processing Magazine*, pp. 27–42, 2003.
- [31] T. Quatieri and A. Oppenheim, "Iterative techniques for minimum phase signal reconstruction from phase or magnitude," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 29, no. 6, pp. 1187–1193, 1981.
- [32] R. McGowan and R. Kuc, "A direct relation between a signal time series and its unwrapped phase," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 30, no. 5, pp. 719–726, 1982.
- [33] H. Al-Nashi, "Phase unwrapping of digital signals," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 11, pp. 1693–1702, 1989.
- [34] Z. Karam and A. Oppenheim, "Computation of the one-dimensional unwrapped phase," *Int. Conf. on Digital Signal Processing*, pp. 304–307, 2007.
- [35] L. Keel and S. Bhattacharyya, "Root counting, phase unwrapping, stability and stabilization of discrete time systems," *Linear Algebra and its Applications*, vol. 351–352, pp. 501–518, 2002.
- [36] D. Mitrinovic and J. Keckic, "The cauchy method of residues: Theory and applications," *D. Reidel Publishing Company*, 1984.
- [37] Y. Bistritz, "Zero location with respect to the unit circle of discrete-time linear system polynomials," *Proc. of the IEEE*, vol. 72, no. 9, pp. 1131–1142, 1984.
- [38] M. Benidir, "On the root distribution of general polynomials with respect to the unit circle," *Signal Processing*, vol. 53, pp. 75–82, 1996.
- [39] Online, "CMU ARCTIC speech synthesis databases," [http : //www.festvox.org/cmu\\_arctic](http://www.festvox.org/cmu_arctic), 2014.